A systematic construction of representations of quaternionic type

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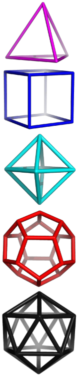
Overview

Platonic Solids



Platonic Solid	Group	root system
Tetrahedron	A_3 A_1^3	Cuboctahedron Octahedron
Octahedron Cube	B_3	Cuboctahedron +Octahedron
Icosahedron Dodecahedron	H_3	Icosidodecahedron

- Platonic Solids have been known for millennia

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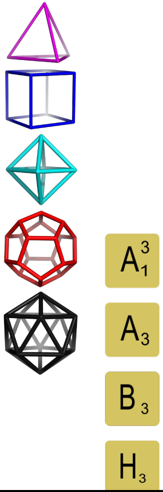

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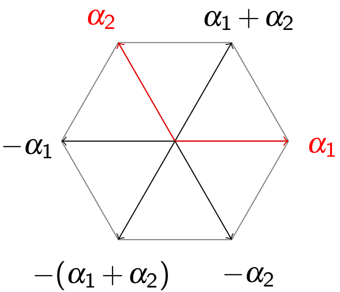
- Platonic Solids have been known for millennia
- Described by Coxeter groups

Root systems



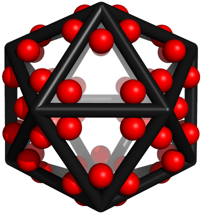
Root system Φ: set of vectors α in a vector space with an inner product such that

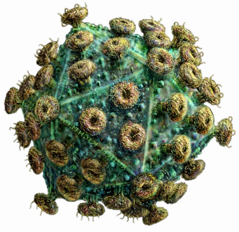
1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$
2. $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

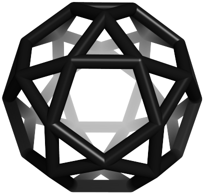
Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination.

reflection/Coxeter groups $\boxed{s_\alpha : v \rightarrow s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha}$

The Icosahedron

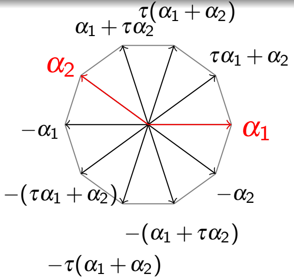


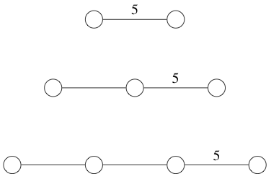




- Rotational icosahedral group is $I = A_5$ of order 60
- Full icosahedral group is H_3 of order 120 (including reflections/inversion); generated by the root system icosidodecahedron

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix}2 & -\tau\\ -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -\tau\\ 0 & -\tau & 2\end{pmatrix}$$

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$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5

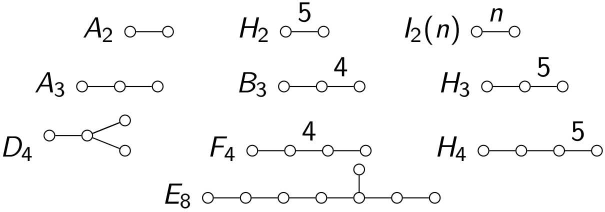
linear combinations now in the extended integer ring

$\boxed{\mathbb{Z}[\tau]=\{a+\tau b | a,b\in\mathbb{Z}\}}$ golden ratio $\boxed{\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}}$

$\boxed{x^2=x+1}$ $\boxed{\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}}$ $\boxed{\tau+\sigma=1, \tau\sigma=-1}$

Cartan-Dynkin diagrams

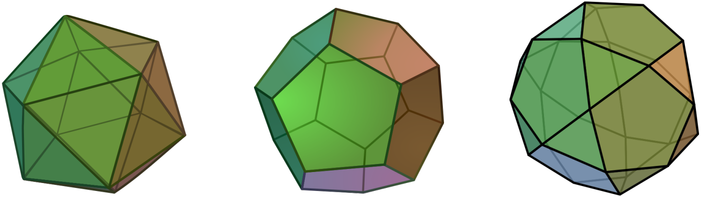
Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal i.e. angle $\frac{\pi}{2}$, simple link = roots at angle $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.



H_3 – the icosahedral group 

$$\boxed{\alpha_1 = (0,1,0),\ \alpha_2 = -\frac{1}{2}(-\sigma,1,\tau),\ \alpha_3 = (0,0,1)}$$

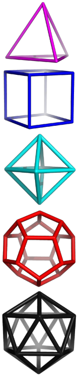
$$\boxed{T_5 = (\tau,-1,0),\ T_3 = (\tau,0,\sigma),\ T_2 = (1,0,0)}$$



Icosahedron, Dodecahedron, Icosidodecahedron (H_3 root system)

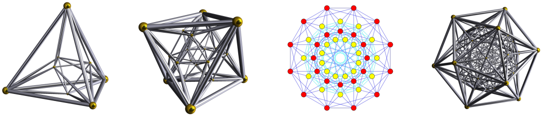
Overview

Platonic Solids



A_1^3	A_1^4
A_3	D_4
B_3	F_4
H_3	H_4

- Concatenating reflections gives Clifford spinors (binary polyhedral groups)
- These induce 4D root systems
  $R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow$
  $R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- 4D analogues of the Platonic Solids and give rise to 4D Coxeter groups



Clifford Algebra and orthogonal transformations

- Geometric Product for two vectors $\boxed{ab \equiv a \cdot b + a \wedge b}$
- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in n is given by $\boxed{a' = a - 2(a \cdot n)n = -nan}$ (n and $-n$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal transformation can be written as successive reflections, which are doubly covered by Clifford versors/pinors A

$$\boxed{x' = \pm n_1 n_2 \ldots n_k x n_k \ldots n_2 n_1} =: \pm A x \tilde{A}$$

Clifford Algebra of 3D

- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

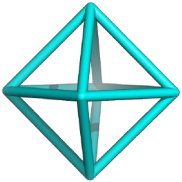
- We can multiply together root vectors in this algebra $\alpha_i\alpha_j \ldots$
- A general element has 8 components, even products (rotations/spinors) have four components:

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

Spinors from reflections



- The 6 roots $(\pm 1, 0, 0)$ and permutations in $A_1 \times A_1 \times A_1$ generate 8 spinors:
- $\boxed{\pm e_1,\ \pm e_2,\ \pm e_3}$ give the 8 spinors $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a discrete spinor group isomorphic to the quaternion group Q.
- As 4D vectors these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 roots of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

Induction Theorem – root systems

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- Check axioms:

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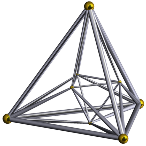
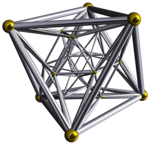
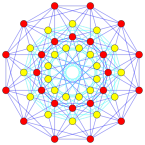
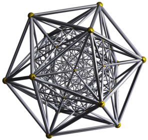
  1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \;\forall\, \alpha \in \Phi}$
  2. $\boxed{s_\alpha \Phi = \Phi \;\forall\, \alpha \in \Phi}$
- Proof: 1. R and $-R$ are in a spinor group by construction (double cover of orthogonal transformations), 2. closure under reflections is guaranteed by the closure property of the spinor group (with a twist: $-R_1\tilde{R}_2R_1$)

Spinors from reflections

- Symmetry groups of the Platonic Solids:
- The 6/12/18/30 reflections in $A_1 \times A_1 \times A_1/A_3/B_3/H_3$ generate 8/24/48/120 spinors.
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).

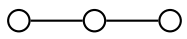
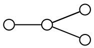
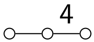
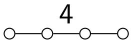
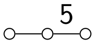
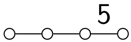
Spinors and Polytopes

- Can reinterpret spinors in $\mathbb{R}^3$ as vectors in $\mathbb{R}^4$
- Give (exceptional) root systems (D_4, F_4, H_4)
- They constitute the vertices of the 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell
- These are 4D analogues of the Platonic Solids. Strange symmetries better understood in terms of 3D spinors

Root systems in three and four dimensions

The spinors from the reflections in the rank-3 Coxeter group via the geometric product are the binary polyhedral groups Q, $2T$, $2O$ and $2I$, which generate (mostly exceptional) rank-4 groups, but not known why, and why the 'mysterious symmetries'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ$	Q	$A_1 \times A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ \quad \circ$
A_3		$2T$	D_4	
B_3		$2O$	F_4	
H_3		$2I$	H_4	

Overview

Quaternion groups via the geometric product

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are the Lipschitz units,the quaternion group in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are the Hurwitz units, the binary tetrahedral group of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form the binary octahedral group of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm\tau, \pm 1, \pm\sigma)$ and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.
- The unit spinors $\{1; e_2e_3; e_3e_1; e_1e_2\}$ of $\mathrm{Cl}(3)$ are isomorphic to the quaternion algebra $\mathbb{H}$.

H_4 from icosahedral spinors

- The H_3 root system has 30 roots e.g. simple roots $\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3)$ and $\alpha_3 = e_3$.
- The subgroup of rotations is A_5 of order 60
- These are doubly covered by 120 spinors of the form $\alpha_1\alpha_2 = -\frac{1}{2}(1-(\tau-1)e_1e_2 + \tau e_2e_3)$, $\alpha_1\alpha_3 = e_2e_3$ and $\alpha_2\alpha_3 = -\frac{1}{2}(\tau-(\tau-1)e_3e_1 + e_2e_3)$.
- As a set of vectors in 4D, they are

$$(\pm 1, 0, 0, 0) \text{ (8 permutations) },\ \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ (16 permutations)}$$

$$\frac{1}{2}(0, \pm 1, \pm\sigma, \pm\tau) \text{ (96 even permutations) },$$

which are precisely the 120 roots of the H_4 root system.

Systematic construction of the polyhedral groups

- Multiplying together root vectors in the Clifford algebra gave a systematic way of constructing the binary polyhedral groups as 3D spinors = quaternions.
- The 6/12/18/30 roots in $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$ generate 8/24/48/120 spinors.
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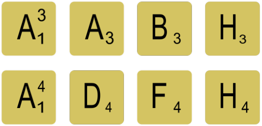
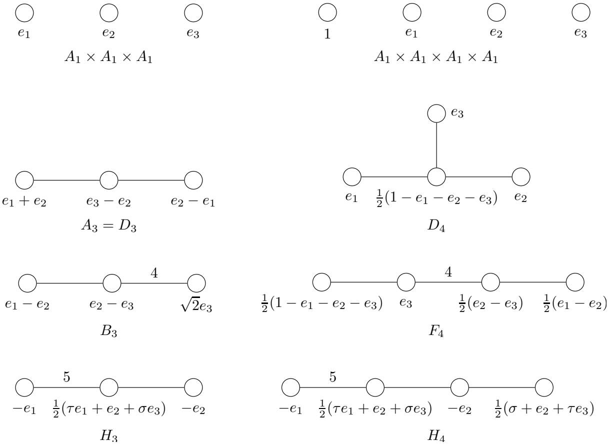


Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8, D_4, F_4 and H_4 have representations in terms of quaternions
- Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm\tau, \pm 1, \pm\sigma)$
- Seen as remarkable that the subset of the 30 pure quaternions is a realisation of H_3 (a sub-root system)
- Similarly, B_3 and $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions
- Clifford provides a much simpler geometric explanation

Quaternionic representations in the literature



Pure quaternions = Hodge dualised root vectors

Quaternions = spinors

Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar I
- e.g. does not work for the tetrahedral group A_3, but $A_3 \to D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1\alpha_2$ and $R_2 = \alpha_2\alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- Sandwiching is often seen as particularly nice feature of the quaternions giving rotations
- This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D
- However, the root system construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is self-dual
- Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$
- Recently constructed E_8 from the 240 pinors doubly covering 120 elements of H_3 in $2^3 = 8$-dimensional 3D Clifford algebra

Overview

Polyhedral groups as multivector groups

Group	Discrete subgroup	Order	Action Mechanism
$SO(3)$	rotational (chiral)	$\lvert G\rvert$	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$2\lvert G\rvert$	$x \rightarrow \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$2\lvert G\rvert$	$(R_1, R_2) \rightarrow R_1 R_2$
$\mathrm{Pin}(3)$	pinory (?)	$4\lvert G\rvert$	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the chiral icosahedral group has 60 elements, encoded in GA by 120 rotors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements doubly covered by 240 pinors

Some Group Theory: chiral, full, binary, pin

- Easy to calculate conjugacy classes etc of versors in GA
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): 1, $1'$, $1''$, 2_s, $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, $1'$, 2, 2_s, $2'_s$, 3, $3'$, 4_s
- icosahedral (60/120): 1, 2_s, $2'_s$, 3, $\bar{3}$, 4, 4_s, 5, 6_s
- All binary are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with Trinities and the McKay correspondence

Representations from Clifford multivector groups

- The usual picture of orthogonal transformations on an n-dimensional vector space is via $n \times n$ matrices acting on vectors, immediately making connections with representations = matrices satisfying the group multiplication laws.
- Easy to construct representations with (s)pinors in the 2^n-dimensional Clifford algebra as reshuffling components.
- Spinors leave the original n-dimensional vector space invariant, reshuffle the components of the vector.
- But can also consider various representation matrices acting on different subspaces of the Clifford algebra.

Representations from Clifford multivector groups – trivial, parity, rotation representations

- The scalar subspace is one-dimensional. $\tilde{R}1R = \tilde{R}R = 1$ gives the trivial representation, and likewise pinors A give the parity.
- The double-sided action $\tilde{R}xR$ of spinors R on a vector x in the n-dimensional vector space gives an $n \times n$-dimensional representation, which is just the usual rotation matrices.
- E.g. e_1e_2 acting on $x = x_1e_1 + x_2e_2 + x_3e_3$ gives $e_2e_1xe_1e_2 = -x_1e_1 - x_2e_2 + x_3e_3$ which could also be expressed as $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ x_3 \end{pmatrix}$
- If the spinors were acting as $Rx\tilde{R}$ would give a potentially different representation.

Characters, their norm, and the Frobenius-Schur indicator

- Similarity transformed representations are also good representations, but are not fundamentally different: they are equivalent.
- So want a measure for a representation that is invariant under similarity transformations, e.g. the trace aka the character χ of a matrix
- A class function i.e. the same within a conjugacy class because of the cyclicity of the trace
- The character norm $||\chi||^2 := \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2$
- The Frobenius-Schur indicator $\nu := \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$

Real representations of real, complex, and quaternionic type

- $||\chi||^2 = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 1$: representation of real type
- $||\chi||^2 = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 2$: representation of complex type
- $||\chi||^2 = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 4$: representation of quaternionic type
- Theorem: A complex representation is irreducible if and only if $||\chi||^2 = 1$.
- Theorem: A real representation is irreducible if and only if $\boxed{||\chi||^2 + \nu(\chi) = 2}$, e.g. $4 - 2 = 2$ or $1 + 1 = 2$.

Representations from Clifford multivector groups – 8×8 and 4×4 (whole algebra / even subalgebra)

- Rather than restricting oneself to the n-dimensional vector space, one can also define representations by $2^n \times 2^n$-matrices acting on the whole Clifford algebra, i.e. any element acting on an arbitrary elemtent, e.g. here 8×8.
- Likewise, one can define $2^{(n-1)} \times 2^{(n-1)}$-dimensional spinor representations as acting on the even subalgebra.
- 3D spinors have components in $(1, e_1e_2, e_2e_3, e_3e_1)$, multiplication with another spinor e.g. e_1e_2 will reshuffle these components $(e_1e_2, -1, \text{-}e_3e_1, e_2e_3)$
- This reshuffling can therefore be described by a 4×4-matrix.

4×4 – explicit example: A_1^3

- E.g. $\boxed{\pm e_1,\ \pm e_2,\ \pm e_3}$ give the 8 spinors $\boxed{\pm 1, \pm e_1e_2, \pm e_2e_3, \pm e_3e_1}$, or $(\pm 1, 0, 0, 0)$ (8 permutations)
- $||\chi||^2 = 32/8 = 4$, $\nu = -2$ and $||\chi||^2+\nu = 2$ i.e. real irreducible of quaternionic type

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Character table of Q

Q	1	-1	$\pm e_1 e_2$	$\pm e_2 e_3$	$\pm e_3 e_1$
1	1	1	1	1	1
$1'$	1	1	-1	-1	1
$1''$	1	1	-1	1	-1
$1'''$	1	1	1	-1	-1
4_H	4	-4	0	0	0

4×4 – explicit example: A_3

- As a set of vectors in 4D, they are $(\pm 1, 0, 0, 0)$ (8 permutations) , $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ (16 permutations)
- Conjugacy classes: $1 \cdot 4^2 + 1 \cdot (-4)^2 + 6 \cdot 0^2 + 8 \cdot 2^2 + 8 \cdot (-2)^2 = 32 + 32 + 32 = 96$
- $||\chi||^2 = 96/24 = 4$, $\nu = -2$ and $||\chi||^2 + \nu = 2$ i.e. real irreducible of quaternionic type.

3×3 – explicit example: H_3

- Icosahedral spinors are

$$(\pm 1, 0, 0, 0) \text{ (8 permutations) }, \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ (16 permutations)}$$

$$\frac{1}{2}(0, \pm 1, \pm \sigma, \pm \tau) \text{ (96 even permutations) },$$

- E.g. the rotation matrices corresponding to $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ via $\tilde{R}xR$ are

$$\frac{1}{2}\begin{pmatrix} \tau & \tau-1 & -1 \\ 1-\tau & -1 & -\tau \\ -1 & \tau & 1-\tau \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} \tau & 1-\tau & -1 \\ 1-\tau & 1 & -\tau \\ 1 & \tau & \tau-1 \end{pmatrix}.$$

The characters $\chi(g)$ are obviously 0 and τ

- $||\chi||^2 = 120/120 = 1$, $\nu = 1$ and $||\chi||^2 + \nu = 2$ i.e. real irreducible of real type

3×3 – explicit example: H_3 other way

- If the spinors were acting as $Rx\tilde{R}$, then

$$\frac{1}{2}\begin{pmatrix} \tau & 1-\tau & -1 \\ \tau-1 & -1 & \tau \\ -1 & -\tau & 1-\tau \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} \tau & 1-\tau & 1 \\ 1-\tau & 1 & \tau \\ -1 & -\tau & \tau-1 \end{pmatrix},$$

with the same characters as before. Swapping the action of the spinor can change the representation.

4×4 – explicit example: H_3

- Spinors $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ multiplying a generic spinor $R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2$ from the left reshuffles the components (a_1, a_2, a_3, a_0) with the matrices given as

$$\frac{1}{2}\begin{pmatrix} -1 & \tau-1 & 0 & -\tau \\ 1-\tau & -1 & -\tau & 0 \\ 0 & \tau & -1 & \tau-1 \\ \tau & 0 & 1-\tau & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -\tau & 0 & 1-\tau & -1 \\ 0 & -\tau & -1 & \tau-1 \\ \tau-1 & 1 & -\tau & 0 \\ 1 & 1-\tau & 0 & -\tau \end{pmatrix},$$

with characters -2 and -2τ.

4×4 – explicit example H_3: quaternionic type

- 120 4×4 matrices – 9 conjugacy classes, with pairs that have $\pm 2\chi_3$ so gives 4 times that of the 3×3 case
- $|G| \cdot ||\chi||^2 = 1 \cdot 4^2 + 1 \cdot (-4)^2 + 12 \cdot (-2\tau)^2 + 12 \cdot (2\tau)^2 + 12 \cdot (-2\sigma)^2 + 12 \cdot (2\sigma)^2 + 20 \cdot (-2)^2 + 20 \cdot (2)^2 + 30 \cdot 0^2 = 480$
- $||\chi||^2 = 480/120 = 4$, $\nu = -2$ and $||\chi||^2 + \nu = 2$ i.e. real irreducible of quaternionic type

Character table of $I = A_5$

I	1	$20C_3$	$15C_2$	$12C_5$	$12C_5^2$
1	1	1	1	1	1
3	3	0	-1	τ	σ
$\bar{3}$	3	0	-1	σ	τ
4	4	1	0	-1	-1
5	5	-1	1	0	0

Character table of $2I$

I	1	$20C_3$	$30C_2$	$12C_5$	$12C_5^2$	-1	$-20C_3$	$-12C_5$	$-12C_5^2$
1	1	1	1	1	1	1	1	1	1
3	3	0	-1	τ	σ	3	0	τ	σ
$\bar{3}$	3	0	-1	σ	τ	3	0	σ	τ
4	4	1	0	-1	-1	4	1	-1	-1
5	5	-1	1	0	0	5	-1	0	0
2	2	-1	0	$-\sigma$	$-\tau$	-2	1	σ	τ
2	2	-1	0	$-\tau$	$-\sigma$	-2	1	τ	σ
4	4	1	0	-1	-1	-4	-1	1	1
6	6	0	0	1	1	-6	0	-1	-1
4_H	4	-2	0	-2τ	-2σ	-4	2	2τ	2σ
$4_{\tilde{H}}$	4	-2	0	-2σ	-2τ	-4	2	2σ	2τ

A general construction of representations of quaternionic type – canonical representations

- It had so far been overlooked that there is a systematic construction of representations of quaternionic type for 3D polyhedral groups
- This is simply due to the fact that the spinors in 3D provide a realisation of the quaternions
- Therefore spinors provide 4x4 representations of quaternionic type for all (though limited number of) possible groups
- However, they are canonical for a choice of 3D simple roots, i.e. there is a preferred amongst all similarity transformed versions
- These simple roots also determine the 3x3 rotation matrices and their reversed representations in a similar canonical way

Characters in general

- For a general spinor $R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2$ one has 3D character $\chi = 3a_0^2 - a_1^2 - a_2^2 - a_3^2$ and representation

$$\frac{1}{2}\begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 - a_1^2 + a_2^2 - a_3^2 & -2a_0a_1 + 2a_2a_3 \\ -2a_0a_2 + 2a_1a_3 & 2a_0a_1 + 2a_2a_3 & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}$$

- and the 4D rep and character are

$$\begin{pmatrix} a_0 & a_3 & -a_2 & a_1 \\ -a_3 & a_0 & a_1 & a_2 \\ a_2 & -a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_0 \end{pmatrix} \text{ and } \chi = 4a_0.$$

- Characters of the representations are all determined by the spinor!

Overview

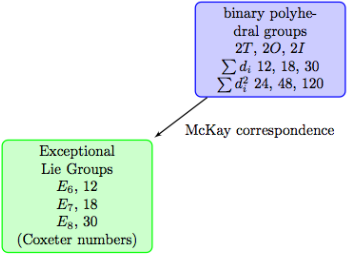
Conclusions

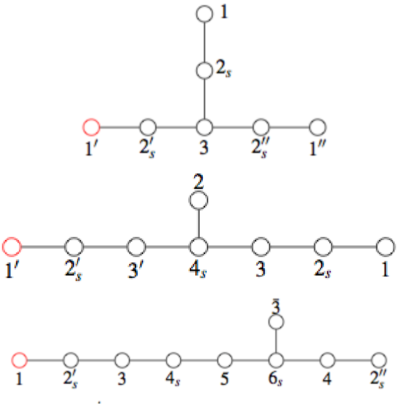
- General construction of 4D root systems from 3D root systems – connections with McKay correspondence, Trinities, Moonshine etc
- Construction systematically and canonically gives representations of 4D root systems and 3D root systems in terms of (pure) quaternions
- Construction systematically and canonically gives construction of the polyhedral groups and their representations, in particular trivial, rotation and spinor representations of quaternionic type with relations among them and their characters

Arnold's Trinities

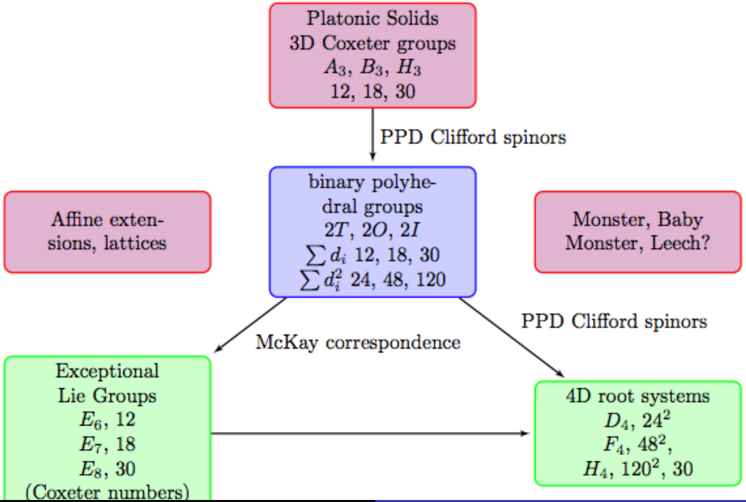
- Arnold noticed that often real, complex and quaternionic versions of a theory are remarkably similar
- Trinities $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$, $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$, the Möbius/Hopf bundles $(S^1 \to S^1, S^4 \to S^2, S^7 \to S^4)$, (E_6, E_7, E_8)
- New connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) (and (E_6, E_7, E_8)!) via my Clifford spinor construction
- Also $(24, 48, 120)$, binary polyhedral groups $(2T, 2O, 2I)$ and $(12, 18, 30)$ (see McKay correspondence)

The McKay Correspondence





The McKay Correspondence

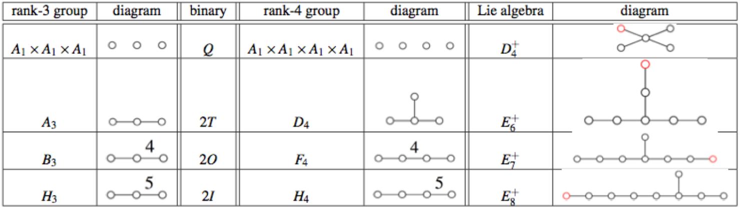


Some Group Theory: chiral, full, binary, pin

- Easy to calculate conjugacy classes etc of versors in GA
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): 1, 1′, 1″, 2_s, $2'_s$, $2''_s$, 3
- octahedral (24/48): 1, 1′, 2, 2_s, $2'_s$, 3, 3′, 4_s
- icosahedral (60/120): 1, 2_s, $2'_s$, 3, $\bar{3}$, 4, 4_s, 5, 6_s
- All binary are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with Trinities and the McKay correspondence

The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the cyclic and dicyclic groups are in correspondence with A_n and D_n, e.g. the quaternion group Q and D_4^+. So McKay correspondence not just a trinity but ADE-classification. We also have $I_2(n)$ on top of the trinity (A_3, B_3, H_3)



rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$		D_4^+	
A_3		$2T$	D_4		E_6^+	
B_3	4	$2O$	F_4	4	E_7^+	
H_3	5	$2I$	H_4	5	E_8^+	